

# SMALL SPECTRAL RADIUS AND PERCOLATION CONSTANTS ON NON-AMENABLE CAYLEY GRAPHS

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**ABSTRACT.** Motivated by the Benjamini-Schramm non-uniqueness of percolation conjecture we study the following question. For a given finitely generated non-amenable group  $\Gamma$ , does there exist a generating set  $S$  such that the Cayley graph  $(\Gamma, S)$ , without loops and multiple edges, has non-unique percolation, i.e.,  $p_c(\Gamma, S) < p_u(\Gamma, S)$ ? We show that this is true if  $\Gamma$  contains an infinite normal subgroup  $N$  such that  $\Gamma/N$  is non-amenable. Moreover for any finitely generated group  $G$  containing  $\Gamma$  there exists a generating set  $S'$  of  $G$  such that  $p_c(G, S') < p_u(G, S')$ . In particular this applies to free Burnside groups  $B(n, p)$  with  $n \geq 2, p \geq 665$ . We also explore how various non-amenability numerics, such as the isoperimetric constant and the spectral radius, behave on various growing generating sets in the group.

## 1. INTRODUCTION

Let  $\Gamma$  be an infinite finitely generated group and let  $S$  be a finite symmetric generating set in  $\Gamma$ . The *isoperimetric constant* (or the *Cheeger constant*) of  $\Gamma$  with respect to  $S$  is

$$\phi(\Gamma, S) = \inf_{F \subset \Gamma} \frac{\sum_{s \in S} |sF \setminus F|}{|F|},$$

where the infimum is taken over all finite subsets  $F$  of  $\Gamma$ . Equivalently we can write

$$\phi(\Gamma, S) = \inf_{F \subset \Gamma} \frac{|\partial_E F|}{|F|},$$

where the infimum is taken over all finite subsets  $F$  of  $\Gamma$  and where  $\partial_E F$  denotes the boundary of  $F$ , i.e., the set of all edges connecting  $F$  to its complement in the Cayley graph of  $\Gamma$  with respect to  $S$ .

The isoperimetric constant  $\phi(\Gamma, S)$  normalized by the size of the generating set is often called the *conductance* constant of  $\Gamma$  with respect

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to  $S$ :

$$h(\Gamma, S) = \frac{1}{|S|} \phi(\Gamma, S).$$

Note that we can also define a different isoperimetric constant  $\phi_V(\Gamma, S)$  by considering the boundary  $\partial_V$  understood as the set of vertices at distance 1 from  $F$ . We have

$$\phi_V(\Gamma, S) \leq \phi(\Gamma, S) \leq |S| \phi_V(\Gamma, S).$$

Let  $\lambda : \Gamma \rightarrow B(l_2\Gamma)$  be the left-regular representation of  $\Gamma$ . The *spectral radius* of  $\Gamma$  with respect to  $S$  is

$$\rho(\Gamma, S) = \frac{1}{|S|} \|\sum_{g \in S} \lambda(g)\|.$$

Then from [12] and [8] we have the following characterization of amenability:

- (1)  $\Gamma$  is amenable,
- (2)  $\rho(\Gamma, S) = 1$ , for some (iff for every) finite generating set  $S \subseteq \Gamma$ ,
- (3)  $\phi(\Gamma, S) = \phi_V(\Gamma, S) = h(\Gamma, S) = 0$ , for some (iff for every) finite generating set  $S \subseteq \Gamma$ .

Note that the conductance constant and the spectral radius are connected by the following inequalities (see [16]):

$$(1) \quad \frac{|S|(1 - \rho(\Gamma, S))}{|S| - 1} \leq h(\Gamma, S) \leq \sqrt{1 - \rho(\Gamma, S)^2}.$$

Equivalently

$$(2) \quad 1 - \frac{h(\Gamma, S)(|S| - 1)}{|S|} \leq \rho(\Gamma, S) \leq \sqrt{1 - h(\Gamma, S)^2}.$$

For  $0 \leq p \leq 1$  consider the *Bernoulli bond percolation* on the Cayley graph  $(\Gamma, S)$ . Namely, an edge  $(g, sg)$  is open with probability  $p$  and closed with probability  $1 - p$ ; connected components of the subgraph spanned by the open edges are called *open clusters*. By Kolmogorov's 0–1 law, the probability of existence of an infinite open cluster is either 0 or 1, and there exists a *critical value*  $p_c$  – the smallest such that an infinite cluster exists almost surely for all  $p > p_c$ . The critical value can also be characterized with the help of the percolation function  $\theta(p)$  defined as the probability that the origin  $e \in \Gamma$  belongs to an infinite open cluster:

$$p_c(\Gamma, S) = \sup\{p : \theta(p) = 0\}.$$

A second critical value can be defined via  $\xi(p)$ , the probability that there exists exactly one infinite open cluster ([10]):

$$p_u(\Gamma, S) = \inf\{p : \xi(p) = 1\}.$$

In general,

$$0 < p_c \leq p_u \leq 1.$$

In the amenable case, the two critical values coincide ([6]), and Benjamini and Schramm conjectured that this property characterizes amenability.

The whole setup and the conjecture can in fact be formulated in the more general situation of locally finite infinite quasi-transitive graphs. In [5], the conjecture was proved for planar graphs. It was also shown that if  $G$  is a locally finite quasi-transitive graph, then there exists a constant  $k = k(G)$  such that the conjecture holds for the product of  $G$  with a regular tree of degree higher than  $k$ . In this paper we will only work with Cayley graphs, and in this case the *Benjamini-Schramm non-uniqueness of percolation conjecture* says the following.

**Conjecture 1** ([4]). *If  $\Gamma$  is a non-amenable group generated by a finite set  $S$  then*

$$p_c(\Gamma, S) < p_u(\Gamma, S).$$

The only non-amenable groups where the non-uniqueness of percolation conjecture is proved for all generating sets are groups with cost greater than 1, as shown by R. Lyons ([14] see also [15]). For the approach to the Benjamini-Schramm conjecture through the theory of measurable equivalence relations see Gaboriau's paper [9].

Benjamini and Schramm showed that the critical value  $p_c$  satisfies the following inequality ([4]):

$$p_c(\Gamma, S) \leq \frac{1}{\phi(\Gamma, S) + 1}.$$

They also proved the following sufficient condition.

**Theorem 2** ([4]). *If*

$$(3) \quad \rho(\Gamma, S) p_c(\Gamma, S) |S| < 1,$$

*then  $p_c(\Gamma, S) < p_u(\Gamma, S)$ .*

Theorem 2 can be used to obtain further sufficient conditions for  $p_c(\Gamma, S) < p_u(\Gamma, S)$ , as for example if combined with the estimate on  $p_c(\Gamma, S)$  in terms of  $\rho(\Gamma, S)$ ,  $|S|$  and the girth of the graph, obtained in [3]. Another corollary of (3) is

**Proposition 3** ([17]). *If  $\rho(\Gamma, S) < \frac{1}{2}$ , then  $p_c(\Gamma, S) < p_u(\Gamma, S)$ .*

Pak and the second author derived from (3) the following weak version of Benjamini-Schramm conjecture for non-amenable groups.

**Theorem 4** ([17]). *For every non-amenable group  $\Gamma$  and any symmetric finite generating set  $S$  there exists a positive integer  $k$  such that  $p_c(\Gamma, S^{(k)}) < p_u(\Gamma, S^{(k)})$ .*

Note that  $S^{(k)}$  stands here for the  $k$ -th power of the set  $S \cup \{e_G\}$  understood as a *multiset*, so that the corresponding Cayley graph has lots of multiple edges. It is thus natural to look for a proof of the same result, but where only “simple” generating sets are allowed. Here *simple* means that the corresponding Cayley graph is a graph without loop or multiple edge. In particular it would be desirable to have Theorem 4 with the multiset  $S^{(k)}$  replaced by the set  $S^k = S \cdot S \dots S \subset \Gamma$ . From now on, by *generating set* we will only mean *simple generating set*.

It was observed already in [17] that if  $\Gamma$  contains a free group on two generators then there exist generating sets  $\{A_l\}_{l \geq 1}$  such that  $\rho(\Gamma, A_l) \rightarrow 0$  and thus for  $l$  big enough we have  $p_c(\Gamma, A_l) < p_u(\Gamma, A_l)$  by Proposition 3.

In view of its relation to the non-unicity of percolation problem, it would be interesting to know whether the spectral radius can always be made arbitrarily small in a finitely generated non-amenable group, in other words, whether non-amenability is equivalent to the following stronger property:

**Definition 5.** Let  $\Gamma$  be a non-amenable group. Suppose that there exists a sequence of finite generating sets  $\{A_l\}_{l \geq 1} \subset \Gamma$  such that

$$\rho(\Gamma, A_l) \rightarrow 0, \text{ as } l \rightarrow \infty.$$

We then say that  $\Gamma$  has *infinitesimally small spectral radius* with respect to the family of generating sets  $\{A_l\}_{l \geq 1}$ .

By Proposition 3, having infinitesimally small spectral radius with respect to generators  $\{A_l\}_{l \geq 1}$  implies that for big  $l$ , there is non-unicity of percolation on Cayley graphs of  $(\Gamma, A_l)$ .

**Question 6.** (1) Does every non-amenable group have infinitesimally small spectral radius with respect to some family of generating sets?

- (2) With respect to some family of the form  $\{S^k\}_{k \geq 1}$ ?
- (3) With respect to  $\{S^k\}_{k \geq 1}$  for any generating set  $S$ ?

In view of Proposition 3 and for the sake of completeness, let us also mention the following related open question:

**Question 7.** Let  $\Gamma$  be a non-amenable group. Does there exist a generating set  $S$  in  $\Gamma$  with  $\rho(\Gamma, S) < 1/2$ ?

Note that if there is an element  $g$  of infinite order in  $\Gamma$ , then the spectral radius can be made arbitrarily close to 1 by adding to any given generating set bigger and bigger powers of  $g$ . In Section 3 (Theorem 16), we show that the spectral radius is bounded away from 1 uniformly on  $S^k$ , as  $k \rightarrow \infty$ , for arbitrary  $S$  or, equivalently (by (2)), that the conductance constant is bounded away from 0 uniformly on  $\{S^k\}$ ,  $k \rightarrow \infty$ . In Section 4 we discuss a new condition equivalent to non-amenability, which implies in particular that in a non-amenable group the isoperimetric constant can be made arbitrarily close to one.

It is easy to check that if a subgroup of a group has infinitesimally small spectral radius then the same property holds for the group. As noted above, groups with free subgroups have infinitesimally small spectral radius. This result is extended to other classes of non-amenable groups in Section 2 below, where we show that the spectral radius goes to 0 on certain sequences of generating sets in the group, and therefore the non-uniqueness of percolation conjecture holds at least on some generating sets, for groups with nontrivial non-amenable quotients (Theorem 10 and Corollary 11 in Section 2). This allows to make the same conclusion for infinite free Burnside groups (Corollary 12 in Section 2) and for direct products  $\Gamma \times \mathbb{Z}/d\mathbb{Z}$  with non-amenable  $\Gamma$  and  $d$  big enough (Corollary 13 in Section 2).

For a finitely generated non-amenable group, the question about the behaviour of the spectral radius on Cayley graphs of  $\Gamma$  with respect to  $S^k$  remains very much open. One sufficient condition for the spectral radius to be infinitesimally small on generators  $S^k$  is Property (RD) of Jolissaint [11].

**Proposition 8** ([17]). *Suppose  $\Gamma$  is a finitely generated non-amenable group with Property (RD). Then  $\Gamma$  has infinitesimally small spectral radius with respect to the sequence  $\{S^k\}_{k \geq 1}$  for any finite symmetric generating set  $S$ .*

In Section 2, Corollary 15, we show that a group  $\Gamma$  has infinitesimally small spectral radius with respect to the sequence  $\{S^k\}_{k \geq 1}$  for a finite generating set  $S$ , if

$$(4) \quad \frac{\rho(\Gamma, S)|S|}{gr(\Gamma, S)} < 1,$$

where  $gr(\Gamma, S)$  denotes the rate of exponential growth of  $\Gamma$  with respect to  $S$ . Observe that the estimate  $p_c(\Gamma, S) \leq gr(\Gamma, S)^{-1}$  combined with the condition (3) implies that (4) is enough to guarantee  $p_c(\Gamma, S^k) < p_u(\Gamma, S^k)$  for big enough  $k$ . Our statement shows that (4) implies in fact a stronger property, that  $\rho(\Gamma, S^k) \rightarrow 0$  when  $k \rightarrow \infty$ .

More generally, for any non-amenable quasi-transitive graph, it would be interesting to know how the spectral radius changes when edges are added to the graph so as to connect all vertices inside bigger and bigger balls.

**Question 9.** Let  $G$  be a quasi-transitive locally finite non-amenable graph. For every  $k \geq 1$ , define  $G_k$  by adding to  $G$  edges connecting any two vertices at distance  $\leq k$  in  $G$ . What is the asymptotics of  $\rho(G_k)$  as  $k \rightarrow \infty$ ? Is it true that  $\rho(G_k) \rightarrow 0$ ?

Along the paper we assume that all  $G$  is generated by a finite symmetric set  $S$ .

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## 2. SPECTRAL RADIUS AND NON-UNICITY OF PERCOLATION

In this Section we investigate the property of infinitesimally small spectral radius and draw conclusions about non-unicity of percolation.

**Theorem 10.** *If  $\Gamma$  contains an infinite normal subgroup  $N$  such that  $\Gamma/N$  is non-amenable, then  $\Gamma$  has infinitesimally small spectral radius.*

*Proof.* Since  $\Gamma/N$  is non-amenable, there exists a generating set  $S$  in  $\Gamma/N$  with  $\rho(\Gamma/N, S) \leq C < 1$ . Thus  $\rho(\Gamma/N, S^{(n)}) = \frac{1}{|S|^n} \|\sum_{g \in S^{(n)}} \lambda(g)\| = (\rho(\Gamma/N, S))^n \leq C^n$ , where  $S^{(n)}$  is the  $n$ -th power of  $S$  understood as multiset. In particular, in  $S^{(n)}$  each element  $g \in S^n$  is included as many times as there exist freely reduced words of length at most  $n$  in the alphabet  $S$  representing  $g$ ; denote this number  $\alpha(g)$ . We will now construct a lift of the generating multiset  $S^{(n)}$  to a simple generating set in  $\Gamma$ , preserving the cardinality and controlling the spectral radius. Begin by choosing a lift  $\hat{S}^n$  of the set  $S^n$  to  $\Gamma$  such that  $|S^n| = |\hat{S}^n|$ . Now for each  $g \in S^n$  choose  $\alpha(g)$  different elements in  $N$ :  $\{h_{g,1}, \dots, h_{g,\alpha(g)}\}$ . Now consider

$$\bar{S}_n = \bigcup_{\omega \in \hat{S}^n} \{h_{g,1}g, h_{g,2}g, \dots, h_{g,\alpha(g)}g\}.$$

The generating set  $\bar{S}_n$  projects onto  $S^{(n)}$  under the canonical projection, and thus we have  $\rho(\hat{S}_n, \Gamma) \leq C^n$  which implies the statement.  $\square$

**Corollary 11.** *Let  $\Gamma$  be a discrete group that contains an infinite normal subgroup  $N$  such that  $\Gamma/N$  is non-amenable. Then there exists a finite set  $S \subset \Gamma$  such that*

$$p_c(\Gamma, S) < p_u(\Gamma, S).$$

Let  $B(n, p)$  be the free Burnside group on  $n$  generators. By [1] the group  $B(n, p)$  is non-amenable for  $n \geq 2, p \geq 665$ . We also have results of [18] that show that for these groups  $B(2n, p) < B(n, p)$  and there exists a canonical quotient map from  $B(2n, p)$  onto  $B(n, p) \times B(n, p)$ .

**Corollary 12.** *Let  $B(n, p)$  be the free Burnside group with  $n \geq 2, p \geq 665$ . Then there exists a finite set  $S \subset B(n, p)$  such that*

$$p_c(B(n, p), S) < p_u(B(n, p), S).$$

As a direct modification of the proof of the Theorem 10 and the Proposition 3 we have the following.

**Corollary 13.** *Let  $\Gamma$  be a non-amenable finitely generated group. Then there exists  $d = d(\Gamma, S)$  such that  $p_c(\Gamma', S') < p_u(\Gamma', S')$  for  $\Gamma' = \Gamma \times \mathbb{Z}/d\mathbb{Z}$  with a generating set  $S'$ .*

In a Cayley graph of a finitely generated group  $\Gamma$  with respect to a generating set  $S$ , denote by  $B_k(\Gamma, S)$  the ball of radius  $k$ ,  $k \in \mathbb{N}$ . Denote by  $gr(\Gamma, S) = \lim_{n \rightarrow \infty} |B_n(\Gamma, S)|^{\frac{1}{n}}$  the rate of exponential growth of  $\Gamma$  with respect to the generating set  $S$ , strictly bigger than 1 for any  $S$  in a non-amenable  $\Gamma$ .

**Lemma 14.** *Let  $\Gamma$  be a group generated by a finite set  $S$ . Then*

$$\rho(\Gamma, S^k) \leq \frac{|S|^k}{|S^k|} (\rho(\Gamma, S))^k,$$

*Proof.* Let  $X$  be a finite subset of  $\Gamma$  and  $A = \sum_{g \in X} \alpha_g \lambda(g) \in \mathbb{C}[\Gamma]$ , then we have

$$\|A\| = \lim_{p \rightarrow \infty} \tau((A^* A)^p)^{\frac{1}{2p}},$$

where  $\tau$  is the standard trace on  $\mathbb{C}[\Gamma]$ , i.e.  $\tau$  is linear functional such that  $\tau(g) = 0$  if  $g \neq 1$  and  $\tau(1) = 1$ . Therefore for every  $\beta_g \in \mathbb{N}$ , in particular if  $\beta_g$  is the multiplicity of the element  $g$  in the multiset  $S^{(k)}$ , we have

$$\begin{aligned} \left\| \sum_{g \in S^k} \lambda(g) \right\| &\leq \left\| \sum_{g \in S^k} \beta_g \lambda(g) \right\| \\ &= \left\| \sum_{g \in S^k} \lambda(g) \right\|. \end{aligned}$$

Therefore we have

$$\begin{aligned}\rho(\Gamma, S^k) &\leq \frac{\rho(\Gamma, S^{(k)})|S|^k}{|S^k|} \\ &\leq \frac{(\rho(\Gamma, S)|S|)^k}{|S^k|}.\end{aligned}$$

□

**Corollary 15.** *Let  $\Gamma$  be a non-amenable group and  $S$  a finite generating set in  $\Gamma$ , and let  $gr(\Gamma, S)$  denote the rate of exponential growth of  $\Gamma$  with respect to  $S$ . Assume that*

$$\frac{\rho(\Gamma, S)|S|}{gr(\Gamma, S)} < 1.$$

*Then  $\Gamma$  has infinitesimally small spectral radius with respect to the family  $\{S^k\}_k$  of generating sets:*

$$\rho(\Gamma, S^k) \rightarrow 0 \text{ when } k \rightarrow \infty.$$

*Proof.* Consider the generating sets  $\{S^k\}_k$ . The cardinality  $|S^k|$  of  $S^k$  is equal to the cardinality of the  $k$ -th ball in the Cayley graph of  $\Gamma$  with respect to  $S$ , therefore the assumption implies that there exists  $k_0 \in \mathbb{N}$  and  $C < 1$  such that for every  $k \geq k_0$  we have

$$\frac{\rho(\Gamma, S)|S|}{\sqrt[k]{|S^k|}} \leq C < 1.$$

Therefore, by Lemma 14,  $\rho(\Gamma, S^k) \rightarrow 0$ , when  $k \rightarrow \infty$ . □

### 3. ASYMPTOTICS OF SPECTRAL RADIUS AND CONDUCTANCE CONSTANT ALONG BALLS

In this section we study the asymptotic behavior of the spectral radius and of the conductance constant on Cayley graphs with respect to the generating sets  $S^k$ ,  $k \geq 1$ , for any  $S$ . Since both constants are related by inequalities (1) and (2), in our case it is sufficient to study only one of them.

**Theorem 16.** *Let  $\Gamma$  be non-amenable group generated by a finite set  $S$ . Then there is a constant  $0 < C < 1$  such that for every  $k \in \mathbb{N}$  we have*

$$\rho(\Gamma, S^k) \leq C.$$

*In terms of conductance constant, there exists some  $0 < C' < 1$  such that*

$$h(\Gamma, S^k) \geq C'$$



for every  $k \in \mathbb{N}$ .

*Proof.* To reach a contradiction, assume that  $\rho(\Gamma, S^k) \rightarrow 1$  on some subsequence, then  $h(\Gamma, S^k) \rightarrow 0$ . In terms of conductance constant this means that for every  $\varepsilon > 0$  there exists  $k \in \mathbb{N}$  and a finite set  $F \subseteq \Gamma$  such that

$$\sum_{s \in B_k} |sF \setminus F| \leq \varepsilon |B_k| |F|,$$

where, as above,  $B_k$  denotes the ball of radius  $k$  in the Cayley graph of  $(\Gamma, S)$ , i.e.,  $B_k = S^k$ . Fix  $\varepsilon$  and let  $F$  and  $k$  are given by inequality above. Consider the function  $f = \sum_{g \in B_k} \chi_{gF}$  and let  $\Lambda_k$  be the sphere of radius  $k$ . Then  $\|f\|_{l_1(\Gamma)} = |F| |B_k|$ . Note that for every  $h \in S$  we have  $|hB_k \setminus B_k| = |B_k \setminus hB_k|$  and  $hB_k \setminus B_k \subseteq \Lambda_{k+1}$  and  $B_k \setminus hB_k \subseteq \Lambda_k$ . Thus for  $h \in S$  we have

$$\begin{aligned} \left\| \sum_{g \in B_k} \chi_{hgF} - \sum_{g \in B_k} \chi_{gF} \right\|_1 &= \left\| \sum_{g \in hB_k} \chi_{gF} - \sum_{g \in B_k} \chi_{gF} \right\|_1 \\ &= \left\| \sum_{g \in hB_k \setminus B_k} \chi_{gF} - \sum_{g \in B_k \setminus hB_k} \chi_{gF} \right\|_1 \\ &= \left\| \sum_{g \in hB_k \setminus B_k} (\chi_{gF} - \chi_F) - \sum_{g \in B_k \setminus hB_k} (\chi_{gF} - \chi_F) \right\|_1 \\ &\leq \sum_{g \in \Lambda_{k+1}} \|\chi_{gF} - \chi_F\|_1 + \sum_{g \in \Lambda_k} \|\chi_{gF} - \chi_F\|_1 \\ &= \sum_{g \in \Lambda_{k+1}} |gF \Delta F| + \sum_{g \in \Lambda_k} |gF \Delta F| \\ &\leq \sum_{g \in B_{k+1}} |gF \Delta F| \\ &\leq 2\varepsilon |B_{k+1}| |F| \\ &\leq 2\varepsilon \frac{|B_{k+1}|}{|B_k|} \|f\|_1 \\ &\leq 2\varepsilon |S| \|f\|_1. \end{aligned}$$

Normalizing  $f$  we obtain a positive function  $f \in l_1(\Gamma)$  with  $\|f\|_1 = 1$  and  $\|hf - f\| \leq \varepsilon'$  for every  $h \in S$ . Thus  $\Gamma$  is amenable.  $\square$

#### 4. ISOPERIMETRIC CONSTANTS OF NON-AMENABLE GROUPS

In this section we show a new characterization of amenability that can be viewed as positive evidence towards the conjecture that every non-amenable group has infinitesimally small spectral radius. As an

application we also have a new estimate on the isoperimetric constants of the group.

We will need the following lemma which is well known (see e.g. Proposition 11.5 in [2]), but we include it for completeness.

**Lemma 17.** *Let  $\pi : \Gamma \rightarrow B(H)$  be a unitary representation of a discrete group  $\Gamma$ . Suppose that there exists a unit vector  $\xi \in H$  such that  $\|\pi(g)\xi - \xi\| \leq C < \sqrt{2}$  for every  $g \in \Gamma$ . Then  $\pi$  has an invariant vector.*

*Proof.* Note that

$$\begin{aligned} \operatorname{Re}(\langle \pi(g)\xi, \xi \rangle) &= 1 - \frac{1}{2}\|\pi(g)\xi - \xi\|^2 \\ &\geq 1 - \frac{C^2}{2} = C' > 0. \end{aligned}$$

Let  $V = \overline{\operatorname{conv}\{\pi(g)\xi : g \in \Gamma\}}$  then  $V$  is  $\pi(\Gamma)$ -invariant and

$$\operatorname{Re}(\langle \theta, \xi \rangle) \geq C' \text{ for every } \theta \in V.$$

Let  $\nu \in V$  be the unique element of  $V$  that has minimal norm, then  $\operatorname{Re}(\langle \nu, \xi \rangle) \geq C'$  and  $\nu \neq 0$ . Since  $\pi$  is a unitary representation,  $\nu$  is invariant under the action of  $\pi(\Gamma)$ .  $\square$

**Theorem 18.** *A finitely generated group  $\Gamma$  is amenable if and only if there exists a constant  $C < 2$  such that for every finite set  $S \subset \Gamma$  there exists a finite set  $F \subset \Gamma$  such that*

$$|sF\Delta F| \leq C|F|, \text{ for every } s \in S.$$

*Proof.* The existence of  $C \leq 2$  that satisfy the condition of the theorem for amenable group  $\Gamma$  follows from Følner's criteria.

To prove the converse fix a finite set  $S$  and let  $F$  be a finite set of  $\Gamma$  such that

$$|sF\Delta F| \leq C|F|, \text{ for every } s \in S.$$

Consider  $\xi_F = \frac{1}{\sqrt{|F|}}\chi_F$ , we have  $\|\lambda(s)\xi_F - \xi_F\| \leq \sqrt{C}$  for every  $s \in S$ .

Let  $S_i$  be an increasing sequence of sets in  $\Gamma$  with  $\Gamma = \cup S_i$  and let  $\lambda_\omega : G \rightarrow B(l_2(\Gamma)^\omega)$  be an ultra-limit of the left-regular representation acting on an ultra power of the Hilbert space  $l_2(\Gamma)$ . Then for the vector  $\xi = (\xi_{F_i})_{i \in \mathbb{N}}$  we have that  $\|\lambda_\omega(g)\xi - \xi\| \leq \sqrt{C}$  for every  $g \in G$ . By Lemma 17 we have that  $\lambda_\omega$  has an invariant vector. Thus  $\lambda$  has a sequence of almost invariant vectors, therefore  $\Gamma$  is amenable.  $\square$

As a direct application of the Theorem 18 we have the following corollary.

**Corollary 19.** *Let  $\Gamma$  be a non-amenable group then for every  $\varepsilon > 0$  there exists a finite set  $S \subset \Gamma$  such that*

$$\phi(\Gamma, S) \geq 1 - \varepsilon.$$

**Remark 20.** If the condition equivalent to amenability in Theorem 18 could be strengthened to say that there exists a constant  $C < 2$  such that for every finite set  $S \subset \Gamma$  there exists a finite set  $F \subset \Gamma$  such that

$$\sum_{s \in S} |sF\Delta F| \leq C|F||S|,$$

then it would imply that for every non-amenable group the conductance constant is arbitrary close to 1 on some finite sets and thus the group has infinitesimally small spectral radius, by Mohar's inequalities.

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